CHAPTER 7  A Tale of Two students

Our knowledge is a torch of smoky pine
That lights the pathway but one step ahead
Across a void of mystery and dread.

– George Santana,
O World, Thou Choosest Not the Better Parts

7.1 Introduction

The phenomenon of divergence occurs in classrooms of undergraduate students enrolled in remedial mathematics courses, as well as in the classrooms of elementary age students and students in the middle grades. In order to better understand this repeating pattern which results in success for some and failure for others, it is not enough to document the existence of the phenomenon, but to examine possible causes of the divergence. It was hypothesized that successful students construct, organize, and reconstruct their knowledge in qualitatively different ways than do students who are least successful. These processes are constrained by a student’s initial perception(s) and the categorization of those perceptions which cue selection and retrieval of a schema that directs subsequent actions and thoughts. How knowledge is structured and organized determines the extent to which a student is able to think flexibly. The inability to think flexibly leads to the fragmentation in students’ strategies and the resulting divergence of performance, both quantitatively and qualitatively, between those who succeed and those whose who do not. This divergence of performance has been documented in the preceding chapter.

The focus now shifts to examine more closely the strategies used by students who think flexibly and those who do not. To address the main research question of whether students who are most successful construct, organize, and restructure their knowledge in ways that are qualitatively different from those who are least successful, two students’ processes of constructing their cognitive collages of conceptual structures are examined. Two students who are representative of the extremes of the students who participated in the study are profiled. These students represent subjects from the top 15% and the bottom 15% of the class, based on their responses on the post-course test and the final exams (multiple choice and open response). A brief descrip-
tion of each student’s background is followed by an analysis of each student’s mathematical growth during the semester, based on the data of their pre- and post-tests, their work during the semester, and interviews. The qualitatively different strategies used by each student are described within the theoretical framework. The second main thesis question is addressed: “Do successful students construct, organize, and reconstruct their knowledge in ways that are qualitatively different from the processes utilized by those least successful?” The results of the main study presented in this chapter are interpreted using the theoretical framework set forth in Chapter 3. Data are triangulated with other data collected during the semester and are further analyzed in the next chapter, using students’ concept maps and schematic diagrams of those concept maps.

7.2 Perceptions and Strategies

Let me summarize the bits and pieces of knowledge that have been assembled for two students: MC (S2), a student in the most successful group and SK (S23), a student in the group of least successful students. Gradually, as more and more bits and pieces of knowledge are presented, the initial cognitive collages of these students are restructured into more refined, stable cognitive collages which are used as evidence in support of the thesis and to address the main research question: Do students who are more successful construct, organize, and restructure knowledge in ways that are qualitatively different from those least successful?

MC’s ambition is to be an illustrator and is planning to major in graphic design. He has a look of curiosity about him as he enters class, warily during the first few weeks, with cautious optimism by mid-term, and with genuine pleasure by the end of the semester. His natural inclination to put himself wholeheartedly into whatever task he has set for himself is contagious and becomes more evident as his confidence in his ability to do mathematics grows. Students who work with MC develop a comradeship and support each other’s efforts to succeed. MC had three years of mathematics in high school: Algebra I, Geometry, and Algebra II. He took no mathematics course his senior year. After graduating from high school, he enrolled at the community college which was the site of this research. MC tested into an Arithmetic class. He completed that individualized course, followed by the individualized, self-paced three-part Introductory Algebra course in the Math Lab. He completed all three components
of the Introductory Algebra course successfully and was now enrolled in the regular sixteen-week Intermediate Algebra course. MC maintains that the arithmetic and Introductory Algebra courses were a review of mathematics he had learned previously in high school—most of which he readily admits he was unable to remember. The Intermediate Algebra course was the first course in which he used the graphing calculator. He had never experienced mathematics taught using non-traditional materials and reform instructional practices. On the pre-course attitude survey, he reported that he attended the previous mathematics course regularly, and that he had spent one to three hours per week outside of class on homework. He felt that his ability to interpret notation was somewhat good; his ability to interpret and analyze data fair; his willingness to attempt a problem somewhat poor; and his ability to solve a problem very poor. He believed mathematics was mostly facts and procedures to be memorized.

SK wants to be an elementary grade school teacher (K–3). She is a recent high school graduate and also had three years of mathematics in high school, taking the Algebra II course her senior year. She tested into the Introductory Algebra course and elected to take the individualized three-component, self-paced Introductory Algebra course in the Math Lab, rather than the one-semester classroom-based course. Having successfully completed the three Introductory Algebra components during the previous semester, she was now enrolled in Intermediate Algebra. Like MC, SK reported she had never used the graphing calculator before this class, except for adding, subtracting, etc. On her pre-course student information survey, she indicated that she had attended her previous mathematics class regularly; generally spent three to five hours per week outside of class on homework in her previous class; and rated her ability to interpret mathematical notation and symbols somewhat poor. She considered her ability to interpret and analyze data, her willingness to attempt to solve a problem not seen before, and her ability to solve a problem not seen previously was also somewhat poor. SK was firmly convinced that “there was only one way to learn and teach math” and that “math was about doing a lot of the same problems in order to have an understanding of what you were learning.” She thought that “this work was tedious and boring.”
7.2.1 MC and SK: Ability to Interpret Ambiguous Notation

The performances of both students on the pre-test suggests that MC is more flexible in his thinking initially. He demonstrates an ability to reverse his train of thought [P3 and P1] when given an arithmetic context and is able to find the output for a function using a graph when no rule is stated [P8]. SK is able to square a negative number. By the end of the semester, MC is able to answer all but the two conceptual questions involving the minus symbol. SK correctly answers four of fourteen post-course questions, three of which are questions involving arithmetic computations. Even with the aid of the calculator, SK is not fully confident in her answers when asked to square a negative number [P3] and to find the additive inverse of a number squared [P1]. Figure 7.1 summarizes the pre- and post-test responses for both students.

FIGURE 7.1. MC(S2) and SK(S23): Pre- and Post-test Responses

<table>
<thead>
<tr>
<th>Pre &amp; Post Test Question</th>
<th>MC</th>
<th>SK</th>
<th>MC</th>
<th>SK</th>
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<tbody>
<tr>
<td>Pre &amp; Pre</td>
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<tr>
<td>14. Meaning of f(-x)</td>
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<td></td>
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<tr>
<td>13. Meaning of -f(x)</td>
<td></td>
<td></td>
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<td></td>
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<tr>
<td>11. Tables: find g(f(2))</td>
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<tr>
<td>10. Tables: find f(g(2))</td>
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<tr>
<td>12. Graph: (linear) find eq</td>
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<tr>
<td>9. Graph: find x if y(x) = 8</td>
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<tr>
<td>5. Sign of c in (x - c)</td>
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<tr>
<td>4. Meaning of f(x)</td>
<td></td>
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<td></td>
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<tr>
<td>7. Given f, find f(h-1)</td>
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<tr>
<td>6. Given f, find f(-2)</td>
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<tr>
<td>2. Order of operations</td>
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<tr>
<td>1. -( n squared)</td>
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<td></td>
<td></td>
<td></td>
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<tr>
<td>8. Graph: find y(3)</td>
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<td></td>
<td></td>
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<tr>
<td>3. Square of a negative n</td>
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<td></td>
</tr>
<tr>
<td>Correct responses</td>
<td>3</td>
<td>1</td>
<td>12</td>
<td>4</td>
</tr>
</tbody>
</table>

A closer examination of their actual written responses, together with interview data reveals qualitative differences in their thinking and strategies. MC demonstrates some ability to think flexibly at the beginning of the semester as he recognizes the direct and reverse processes in P1 and P3 and acknowledges them as two distinctly different processes. On the pre-test, MC is asked what comes to mind when he is to evaluate $-3^2$ (P1) and to evaluate $(-3)^2$ (P3). His response for P1:

$3 \cdot 3 ;$ I recognize that $-3^2$ means the opposite of $3^2$ and equals $-9$. 
In response to P3, MC writes:

This problem is different than problem #1 because of the parentheses.
This is solved by squaring \(-3\); \(-3\cdot -3 = 9\)

He rates his confidence in the correctness of his answers to each question as 5, on a scale of 1 (I can’t answer the question) to 5 (very confident in my answer). SK on the same two questions writes:

P1: \(-3 \cdot -3 = -9\) and for P3: \((-3)(-3) = 9\)

SK indicates her confidence in the answer for P1 is 3 (somewhat confident in my answer) and for P3 she rates the answer 4 (fairly confident in my answer). There is no indication that she sees any distinction in the process used in P1 and that used in P3. Her replies suggest that she retrieves and implements two different schemas to answer P1. She perceives \(-3\) as a unit and squares the unit, but maintains the minus symbol in front of her answer. She does not seem to have recognized another rule that is in conflict with her operational rule “a number times itself,” namely that a negative times a negative is a positive and that a value squared, unless it is zero, is always positive. On P3, she appears to use the same operational rule—squaring means a number times itself. Her use of parentheses suggests that she uses them because they are given in the original problem but that they have no other significance for her.

7.2.2 MC and SK: Ability to Think Flexibly to Reverse a Direct Process

The pre- and post-test responses of the two students for the two pairs of questions, designed to test students’ ability to reverse a direct process given a table or a graph of one or more functions and no stated rule, were compared. Pre- and post-test questions, P8 and P9, consisted of a graph of a piece-wise linear function with no stated rule. Students were asked to use the graph to answer Questions 8 and 9.
8. Indicate what $y(8) = \quad$ What comes to mind:  
9. If $y(x) = 2$, what is $x$? $\quad$ What comes to mind: 

On the pre-test, MC attempted an answer for P8. He labelled both the $x$- and $y$-axis and circled the number 3 on the $x$-axis and 2 on the $y$-axis. He wrote for P8: I think of $y$ being 2 since $x$ is 3. For P9, MC described what came to mind:  

P9: I think that $x$ can be any number. I would plug it in and try to solve for $y$ if $x$ were given.

MC’s response indicates he is able to deal with the direct process of evaluating a function using a graph but is unable to see P9 as a reversal of the direct process of evaluating a function, given the input. His description also suggests he has a proto-typical concept image of variable: “When I see the variable, $x$, I’m going to solve for the missing variable.” Though he answered pre-test Question 8 correctly, he indicated he was not confident in his answer, selecting a rating of 1 (I don’t know how to answer the question). His initial confidence rating for Question 9 was a 2 (not very confident in my answer).

SK labels the $x$- and $y$-axis, but makes no attempt to answer either Question 8 or Question 9 on the pre-test. She also gives no response to the question: What comes to mind? and rates her confidence at level 1 (I can’t do this problem). Her responses to the same two questions on the post-test demonstrate almost no improvement in her ability to think flexibly or in her competence, even on procedural questions. Confidence ratings for both problems remains at level 1 (I don’t know how to answer the question). For P8, her response confirms that the confidence rating is valid and that, when in doubt, she falls back on something she knows how to do: Given a graph, label the axes. SK wrote:  

P8: Label $x$ and $y$ [which she has done on the graph].

She answers P9 with two questions of her own.  

P9: Is all $y(x)$ equal 2? Does only $x$ equal 2? 

By the end of the semester MC demonstrates improvement in his ability to think flexibly to reverse a process (P8 and P9; P10 and 11). His response for post-test P8 was succinct and confident:
P8: I assume that \( x = 8 \) and found the \( y \) value; \( y(8) = 4 \)

He circles the point on the graph corresponding to 8 on the \( x \)-axis and rates his confidence in the answer as a 4 (fairly confident in my answer). On P9 on the post-test, his confidence rating is again 4 (fairly confident in my answer) and his answer again succinct:

P9: Scale the \( y \) axis to 2 and scale down \( x \) to find value; \( x = 3 \).

MC’s improved flexibility in thinking to reverse a process was also documented in his responses to Questions 10 and 11 on the post-test, using a table representation to evaluate a composition of two functions, \( f \) and \( g \), without a stated rule for either function on the post-test. Pre- and post-test Questions 10 and 11 were as follows:

Consider the following tables for functions \( f \) and \( g \) then answer Questions 10 and 11.

\[
\begin{array}{c|c}
 x & f(x) \\
\hline
 1 & 3 \\
 2 & -1 \\
 3 & 1 \\
 4 & 0 \\
 5 & -2 \\
\end{array}
\quad
\begin{array}{c|c}
 x & g(x) \\
\hline
 -2 & 3 \\
 -1 & 1 \\
 0 & 5 \\
 1 & 2 \\
 2 & 4 \\
\end{array}
\]

10. What is the value of \( f(g(1)) \)? Why?

11. What is the value of \( g(f(5)) \)? Why?

Though he is unable to answer either question on the pre-test, he nevertheless wrote what thoughts came to mind for each question:

P10: I’m really not sure how these two tables relate to one another other than they’re in the same format. 1 comes to mind because it’s opposite of 2.

P11: –1 comes to mind because its opposite of 2 on the table of functions for \( f \).

His answers on the post test to both questions [P10 and P11] indicate he has begun to think proceptually. His confidence in the correctness of his answers has increased from a 1 for both questions on the pre-test (I don’t know how to do this problem) to a 5 for both questions on the post-test (Very confident in my answer). MC’s
responses to post-test P10 and P11 are shown in Figure 7.2. His explanation suggests that he is able to think of \( f(g(1)) \) as an object that is equivalent to \( f(2) \) in Question 10. He appears confident evaluating a composition of two functions even when no rule is stated. MC refers to “the output \( g(-2) = 3 \)”; he is able to think of \( f(5) \) as \(-2\); and writes that \( f(g(1)) \) is equal to \( f(2) \); Look at table values \( f(2) = -1 \) so \( f(g(1)) = -1 \).”

FIGURE 7.2. MC Post-Test P10 & P11: Ability to Think Flexibly

MC’s work suggests that his initial focus of attention is the notation \( f(g(1)) \), which acts as a cognitive unit used to retrieve a schema, which he subsequently unparses. He maintains an awareness of his objective to determine the value of \( f(g(1)) \). An examination of the work of SK reveals a very different initial focus of attention, the cognitive unit \( f(2) \), which appears to cue a schema constrained by her procedural, inflexible thinking. Her work is displayed in Figure 7.3.

FIGURE 7.3. SK Post-test P10 & P11: Ability to Think Flexibly
A comparison of the pre- and post-test responses of MC and SK to conceptual questions[P4, P5, P13, P14] that require no process provides still other examples of answers which are typical of the students in each of the two extremes. These responses of MC and SK are summarized in Table 7.1.

Table 7.1: MC and SK: Flexible Thinking–Interpreting Ambiguous Notation

<table>
<thead>
<tr>
<th>Question #</th>
<th>RESPONSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>MC pre 4:</td>
<td>$g(x)$</td>
</tr>
<tr>
<td>SK pre 4:</td>
<td>$g(x)$</td>
</tr>
<tr>
<td>MC post 4:</td>
<td>$f(x)$</td>
</tr>
<tr>
<td>SK post 4:</td>
<td>$f(x)$</td>
</tr>
<tr>
<td>MC pre 5:</td>
<td>$(x-c)$</td>
</tr>
<tr>
<td>SK pre 5:</td>
<td>$(x-c)$</td>
</tr>
<tr>
<td>MC post 5:</td>
<td>$(x-c)$</td>
</tr>
<tr>
<td>SK post 5:</td>
<td>$(x-c)$</td>
</tr>
<tr>
<td>MC post 13:</td>
<td>$-f(x)$</td>
</tr>
<tr>
<td>SK post 13:</td>
<td>$-f(x)$</td>
</tr>
<tr>
<td>MC post 14:</td>
<td>$f(-x)$</td>
</tr>
<tr>
<td>SK post 14:</td>
<td>$f(-x)$</td>
</tr>
</tbody>
</table>

Growth in students’ ability to think flexibly and recognize the role of context in interpreting the ambiguity of the minus symbol was not as noticeable as the growth in the ability to deal flexibly with function notation, both procedurally and conceptually. Both MC and SK initially interpret $g(x)$ procedurally, interpreting the notation to mean multiplication of $g$ times $x$. By the end of the semester, MC has developed a more flexible way of thinking about the notation $f(x)$ while SK remains at a procedural level of interpretation: “plug in the values you are given for $x$.” MC focuses on the notation and the input/output process: function notation—>output of a function; SK initially thinks function—>function machine—>plug in values. MC interprets the minus symbol in
front of \( f(x) \) as multiplying the output by \(-1\). His response suggests he perceives the answer as being the opposite of the output value and similarly for the input, given the notation \( f(-x) \). SK’s concept image of the output is of a negative value, not of something that has its sign changed, saying that “\( x \) is negative in that function.”

Both students use two different schemas simultaneously. With no cognitive dissonance or conscious awareness that they are doing so, they mentally use the symbol twice—first to indicate that \( c \) is negative, followed by use of the minus symbol as the subtraction operator: “\( c \) will subtract from any number that comes before the -- symbol.” MC’s response to post-test Question 5 provides some additional evidence that he has developed a more flexible way of thinking about variables and has grown in his ability to interpret ambiguous notation. On the pre-test, he perceived \((x-c)\) as indicating that “the value for \( c \) is negative because of the -- sign in front of \( c \). However, he adds, “\( c \) will subtract from any number that comes before the -- symbol,” illustrating the confusion that results when two concept images are retrieved, along with two distinct schemas for interpretation and use of the minus symbol. SK retrieved a different concept image and schema—when you see a minus symbol in front of a letter, change signs and add. Note that she does not answer the question, which suggests once again, that when confronted with a question she can’t answer, she retrieve a default schema that she knows how to implement.

The post-test response of MC to P5 is consistent with his other post-course interpretations of the minus symbol in conceptual questions and provides triangulated evidence of the development of his ability to think more flexibly: “the value of \( c \) is neither because it may be positive or negative. If \( c \) were positive it would become negative and if it were negative it would become positive.” SK repeats the rule she was taught when subtracting algebraically: subtract–change signs and add; a view that has remained unchanged throughout the semester. She still uses the minus symbol twice; once to subtract and as the sign of \( c \), indicating a negative-valued number.

### 7.3 Shaping and Refining the Cognitive Collages of MC and SK

Are two different concept images of MC (S2) and SK (23) beginning to emerge from the bits and pieces of knowledge presented thus far? It should be mentioned that both MC and SK are conscientious students who attended class regularly and worked
very hard to keep up with their assignments. Both were quiet students, yet strong-willed and fiercely determined to complete the course successfully so that they could get on with their lives. An examination of the summarized competency profiles of these two representative students of the extremes is shown in Figure 7.4. Each row represents a category of six questions. The rows are arranged from A, easiest (bottom) to H, hardest (top). The questions in each group are numbered (1–6) and arranged from left, easiest (1) to right, hardest (6). Both category and question orderings are based on the total number of correct responses of the most successful group of students for each category. Observe that the strengths SK demonstrates appear to be of skills associated with quadratic functions [Row E] and of solving systems of equations [Row F]. MC appears to have approximately the same competencies. This area of strength for both MC and SK, indicated in Figure 7.4 by the white rectangle, is examined in greater detail in the following section.

FIGURE 7.4. MC and SK: Competency Summary Profiles

<table>
<thead>
<tr>
<th>Category of Questions</th>
<th>MC (41)</th>
<th>SK (15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>Interpreting Ambiguous Notation</td>
<td>H</td>
</tr>
<tr>
<td>G</td>
<td>MC Final: Solving Equations</td>
<td>G</td>
</tr>
<tr>
<td>F</td>
<td>Solving Systems</td>
<td>F</td>
</tr>
<tr>
<td>E</td>
<td>MC Final: Quadratics: Skills</td>
<td>E</td>
</tr>
<tr>
<td>D</td>
<td>GR: Given a Quadratic Function</td>
<td>D</td>
</tr>
<tr>
<td>C</td>
<td>Using Graphs</td>
<td>C</td>
</tr>
<tr>
<td>B</td>
<td>Using Tables</td>
<td>B</td>
</tr>
<tr>
<td>A</td>
<td>Quad f: Interpretation &amp; Use</td>
<td>A</td>
</tr>
</tbody>
</table>

7.3.1 Perceptions, Cognitive Units, Concept Images, Retrieval of Schemas

A closer analysis of work which indicates MC’s and SK’s understanding of quadratic functions and of linear systems at the time of the final exam provides us with additional bits and pieces of knowledge to assimilate into the growing cognitive collages of both students, which are typical of students in the two groups of extremes they represent. The divergence in performance was hypothesized to be a consequence of qualitative differences in the strategies students use, the way in which they categorize their initial perceptions, and in the way they structure their knowledge. The theoretical framework elaborated in Chapter 3 is used to interpret both two students’ work.
On the open-response final exam, students were asked to solve a problem typically given students in traditional sections of the Intermediate Algebra and/or the subsequent College Algebra course. They were asked to determine an algebraic model of the parabolic path of a projectile and to determine at what time the projectile would hit the ground. The version of the problem used on the final open response exam is:

A toy rocket is projected into the air at an angle. After 6 seconds, the rocket is 87 feet high. After 10 seconds, the rocket is 123 feet high. After one-half minute, the rocket is 63 feet high.

a. The model for the rocket’s motion is \( h = at^2 + bt + c \) where \( h \) is the height in feet of the rocket after \( t \) seconds. Using the given information, find the values for \( a \), \( b \), and \( c \) so the function models the situation. Briefly explain what you did.

b. Approximate how long it will take for the rocket to hit the ground. Why? Explain how you arrived at your answer.

c. What representation did you choose to investigate this problem? Why?

d. Describe the process you used to find the answers to part a and to part b.

During the semester, problems which required students to determine the parameter values in order to establish an algebraic model for a problem situation, and then to use the model to answer other questions about the situation were a focus of investigation and discussion. The final exam problem was not typical of problems investigated during the semester. During the semester, students were given a set of data and asked to determine the algebraic model. Though they had also studied systems of equations, they had only seen one problem prior to the final exam in which they were asked to solve a system in order to determine parameters. In this instance, students had only the written description of the problem.

Students had several alternative ways in which they could determine the parameter values of a model appropriate for a given situation. They could set up a system of three linear equations in three unknowns and solve the system using matrices on the graphing calculator, or solve the 3 x 3 system algebraically. (three students selected this method). Still other students, having used regression models with actual real world messy data, had realized that traditional textbook problems could be solved simply by entering the ordered pairs into lists, selecting the appropriate regression
model which calculates parameter values appropriate for the problem, enter and graph the algebraic representation, and either use the ROOT [or ZERO] option to find the solution, if \( y = 0 \).

Once the parameters and the algebraic representation of the problem situation were determined, students had several options for determining when the projectile would hit the ground. They could graph the equation and examine the graph to find the x-intercept or they could display table values for input and output, or they could use the TRACE command and approximate the answer. They could also solve the equation algebraically, using the quadratic formula. This problem was rich with options and it was believed that the options individual students selected would reveal something about their thinking.

### 7.3.2 Two paths diverge... the path taken by MC

The work of MC and SK on this problem is compared. Their work was typical of the approach and strategies employed by the other students in their respective groups. MC’s initial focus of attention appears to have been the general algebraic model, which he has circled. This focus of attention is consistent with what he claimed to notice on various post-test questions. An examination of his work suggests that MC’s initial focus of attention cued retrieval of a concept image of quadratic function that includes a notion of the general quadratic equation form, a recognition that a specific model appropriate for the problem conditions is needed and connections to an appropriate schema, having identified that the task was to determine parameter values. He perceives the time/height relationship and records the time and height values as ordered pairs; which he enters in two lists on the calculator. MC’s work is shown in Figure 7.5.
His work and explanations indicate that MC has formed an intelligent partnership with his technology (i.e., he passes control to his tool for certain tasks, then takes back control when it is appropriate, always testing his work against that done by the technology) as described by Jones [1994]. MC graphs the discrete points and examines the resulting plot, having established an appropriate view window. Using the information provided by the plot, MC then selects the quadratic regression option to determine parameter values. He enters his algebraic model and tests it graphically against the plot of the discrete points, saying: “The line falls directly on the ordered pairs.” It should be noted that the number of decimal places used for the parameter values in the model compared to those he initially recorded, [−.5, 17, and 3] was in line with a convention students had used throughout the semester when working on problems which included real world data. The class had agreed to use regression model parameters rounded to three decimal places unless the problem included directions which differed from this convention. MC used the convention consistently, though he occasionally included an additional decimal, as in this problem [−0.5000].
MC demonstrates his ability to interpret the problem, clearly describe his process, and interpret the results of his calculations in a mathematically meaningful way, moving efficiently and appropriately towards his overall objective. He acknowledges his awareness of alternative strategies that might be used in this problem and rejects the algebraic alternative; explaining that he “didn’t feel confident trying to investigate algebraically.” Despite this lack of confidence in his algebraic skills, MC selects an appropriate alternative strategy, using the list, graphing, and table features of the calculator to find an appropriate quadratic regression model and to visualize the time/height relationship. His ability to translate among representations is documented and his work suggests that he has formed mental connections linking the notions of zeros of the function, $x$-intercepts, general quadratic form and the specific algebraic model appropriate to the problem situation.

An interview with MC at mid-term, together with his written self evaluation submitted in his portfolio provides triangulation of his developing ability to interpret and use ambiguous notation, as well as his growth toward proceptual understanding. MC comments:

I’m learning how these algebraic models are set up and what the variables that they contain represent. I’m no longer just blindly solving for $x$, but rather understanding where $x$ (input) came from and how it was found from the data given. Through this kind of learning I have developed an understanding for the use of function notation [$f(x) = output$] and how it replaces the dependent variable, $y$.

He attempts to relate new knowledge to his previously acquired knowledge, claiming:

I have been able to utilize mathematical knowledge that I have gained from previous courses. An example of this is taking my previous skill such as finding slope and applying it to rates of change and from this have moved on to comprehend arithmetic and geometric sequences and then have moved forward even further to understanding linear, exponential, and quadratic models. It’s a good feeling to see things connecting together as I move further along in the text. As I go from investigation to investigation I really see connections in material that are clear and that help establish a solid body of knowledge.

In his final interview of the semester, MC speaks of his understanding of function notation:
I think the most memorable information from this class would be the use and understanding of function notation. A lot of emphasis was put on input and output which really helped me comprehend some algebraic processes such as solving for $x$.

The process of connecting new knowledge to prior knowledge is a goal of his learning. He describes his use of the graphing calculator as a tool for understanding and visualizing mathematics and connection-making:

Another process that was very helpful in understanding algebra (specifically factoring) was using a graph to find the $x$-intercepts to find the zeros of an equation. This is a procedure I had never seen before, but I was able to connect it to my prior knowledge. I found [the graphing calculator] very useful to graph equations to find the number of solutions (finding zeros), and also to find equations when they are unknown (using the graphing calculator as a data process machine).

7.3.3 Two paths diverged...the path taken by SK

The path taken by SK is very different from that taken by MC. An examination of her work on the same final exam problem contributes shape and substance to the cognitive collage of SK. Using the lines and colours of her words, actions, and writings, the picture that emerges presents a stark contrast to the cognitive collage that represents MC. SK’s response to the final exam question is displayed in Figure 7.6.

FIGURE 7.6. Student SK: Final Exam Open Response
Initially, SK focuses on the three time values which she has circled: 6 seconds; 10 seconds; and 30 seconds [a conversion of one-half minute] and notices that she is dealing with a quadratic equation. She ignores the corresponding height values. It is possible that she has some sense of a time/height relationship, but her notion of a time-height relationship appears unconnected with any notion that there is a functional relationship in which time and height values are perceived as ordered pairs and/or input/output values. SK’s cognitive collage possibly includes a concept image of parameter at this point in time, though there is no evidence to support this belief, given her work on this problem. She writes, “I choose these numbers because $h$ equals the height in feet at $t$ seconds. The numbers in front of $t$ equal the seconds.” It seems that, if she has a concept image of parameter, it is a fragmentary collage of bits and pieces of knowledge, organized ineffectively and lacking in interiority. The selection of time values as coefficients of the quadratic equation suggests a compartmentalized cognitive collage in which cognitive dissonances seldom, if ever, arise.

Her initial focus of attention on the three time values, together with the realization that she is dealing with a quadratic equation, $h = at^2 + bt + c$, sets up an inappropriate path-dependent logic characterized by SK’s focus on “getting the answer.” This results in the selection and retrieval of a very different schema from that of MC. Focused on solving a quadratic equation, SK retrieves a schema characterized by her demonstrated tendency to “plug the numbers”—into the equation; into the discriminant; and into the quadratic formula. Her work suggests SK has a very sparse concept image of quadratic function with few connections from the procedures she links to quadratic equations to other cognitive units or concept images, and which is constrained by her inflexible thinking and strategies.

SK’s initial perception of the problem task could be interpreted to indicate that she has an understanding of the problem requirements and a schema by which she can determine the answer. It could be argued that she has recognized the need to create the algebraic model for this problem situation, which requires her to find values for $a$, $b$, and $c$; and that once she has the equation, she solves it to answer the question in part b. However, upon reflection one wonders to what extent she really understands the problem, or whether, uncertain of what to do, she has reverted to using a schema learned previously. Davis [1984] describes the situation in which a student fails to match his
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initial perceptions with a cognitive unit that cues the retrieval of an appropriate schema. This description seems to describe SK:

If no appropriate input can be obtained from the present ‘primitive’ input source, a frame [schema] will typically make a default evaluation...Once an instantiated frame [schema] is judged acceptable, nearly all subsequent information processes used this instantiated frame as a data base. The original primitive data is thereafter ignored. [Davis, 1984, p. 65]

Davis’ description offers an explanation of SK’s behaviour in this instance. Not knowing what to do to find the values of \(a\), \(b\), and \(c\), it seem likely that SK has retrieved a schema she is comfortable using—her cognitive collage of quadratic equations, which consists primarily of memorized procedures (finding the discriminant; use the discriminant value in the quadratic formula to “solve” the equation). One can surmise that this cognitive collage, very refined and very stable, is a schema which is flawed and incomplete, based on the work which documents her failure to take into account that (a) there are no real solutions if the value under the radical is negative; (b) that the quadratic formula is used to determine the values of the independent variable, not the dependent variable; and that (c) the division bar is a grouping symbol indicating that the numerator sum or difference is divided by the denominator, not just the radical.

SK does indeed ignore the original data which does not fit her retrieved schema. Even after sixteen weeks of class investigations and homework assignments which focused on functional relationships and alternative methods of finding parameters using various representations of functions, SK ignores information which provide the data necessary to determine the parameters: the three input/output (time, height) ordered pairs. Her assignment of the time values as coefficients, \(a\), \(b\), and \(c\) suggests she does not yet have a firm understanding of parameters. She demonstrates proficiency in her attention to some details, recognizing the need to have consistent units, changing one-half minute into 30 seconds, and that the model is quadratic. Other details, including the three height values are ignored. Her ability to recall memorized procedures she has associated with quadratic equations is accurate, as is her understanding that the discriminant can be used to simplify quadratic formula computations.

However, the execution of those procedures is flawed. She calculates the discriminant correctly, using incorrect parameter values and fails to interpret the result cor-
rectly. She does not interpret the final result in the context of the problem, choosing instead to state two solutions. Both solution values determined by SK using the quadratic formula are inaccurate, due to her failure to divide correctly. Though aware that the quadratic formula is used to find solutions to a quadratic equation, SK is confused about what variable she is solving for. Once the default quadratic equation schema has been retrieved and actions based on that schema initiated, SK focuses on “getting the answer.” Noticeably lacking is the strategy of checking one’s work against the constraints of the problem, interpreting the results in light of the original problem situation, testing her answers to determine if they make any sense. The connections SK appears to have formed are procedural connections necessary to “get the answer” between the process of finding solutions using the quadratic formula and use of the discriminant to make that process easier. Both procedures are linked to the notion of quadratic equations which includes knowledge that there are, generally, two solutions to a quadratic equation.

Based on her work, one might assume that SK does not know how to solve linear systems using the matrix and/or regression features of the graphing calculator. In fact, of the five problems on the final exam, SK correctly answered four of them. She was able to solve a 2 x 2 inconsistent linear system and a 3 x 3 system, using the matrix features of the graphing calculator. It should be noted, however, that none of the problems she solved correctly were contextual problems. All were written in traditional symbolic form of textbook exercises and all but the 2 x 2 consistent linear system were multiple choice format questions. None of the other final exam linear system problems required more than procedural knowledge. This is another indication of the compartmentalization of SK’s knowledge, in which procedures are linked to a particular concept, in this instance, the use of matrices to solve linear systems, used only when her perception cues the cognitive unit which retrieves the linear systems schema. It seems obvious from her work and explanations that SK did not perceive the toy rocket problem as a system problem. Her original perceptions were classified under the category dealing with quadratic equations, thus failing to recognize that the original information required the retrieval of a strategy for determining parameters based on solving a linear system.
Her initial focus of attention, coupled with her flawed and incomplete construction of her cognitive collages of quadratic equations and systems of equations are underlying causes of her lack of success in this instance. It appears that SK has assembled some bits and pieces of knowledge appropriately, but she is missing other basic pieces. How she has assembled those bits and pieces constrains her ability to construct concept images and cognitive collages that have interiority and which permit meaningful connections. Lacking rich concept images and locked into inflexible thinking, when confronted with situations in which she is unclear what to do, SK retreats to the familiar and defaults to using those procedures she knows.

SK views the graphing calculator as a tool for verifying her calculations. It is used only when she is uncertain which of two calculation procedures to use. In that instance, she just enters what she sees and accepts the calculator answer. Midway through the semester, SK described her feelings about the graphing calculator:

I find the graphing calculator to be very confusing. I feel as if everything is thrown at me at once. I have never used the graphing calculator before this class, and now I find it difficult to adopt to using it. The only thing I can do without too much difficulty is put a table into the calculator. After that I don’t know what to do. A change that would help to improve my learning in this class would be a slower and more thorough explanation of the graphing calculator.

Her growing frustration with the class and with herself increased as the semester passed. She actively resisted assuming responsibility for figuring things out on her own. Provided with handouts that included step by step directions for each procedure introduced during the first eight weeks of the semester—which included views of the screen displays—she never used them. For a student such as SK, learning the calculator procedures in addition to the mathematics she was already struggling with introduced too much cognitive load to cope with. By the end of the semester her feelings were unchanged. In an interview, she said, “Personally I am still overwhelmed by the calculator.” This student remains firmly convinced that

I need that type of concrete repetitious work. In my past math classes I was given a book where there were definitions and formulas to follow. With any type of problem I need to have a step by step process to follow. I have trouble deciding what kind of function it is, or what should go where. For example, I still cannot tell the difference between a linear, quadratic, and exponential model.
She was adamant that there was only one way she could learn math—her way. Perhaps SK was right, though the fact that if that method really worked, she wouldn’t be taking remedial mathematics courses escaped her. Despite the inadequacies of the instructional methods she had experienced previously, and her great efforts and time commitment, she was unwilling to change her beliefs or to try alternatives. Her resistance grew more pronounced during the semester. When the end of the course evaluations were compared with the pre-course responses, SK was one of only two students, both in the low group of extremes, who, not surprisingly, had a more negative attitude that when she enrolled in the course.

7.4 And they will differ...as syllable from sound

The cognitive collages of the two students who are representative of the extremes, the most successful and the least successful, provide detailed evidence of the divergence that occurs over a sixteen week semester. This divergence is much starker than imagined and is portrayed, not in the nuances of soft pastel colours indicative of slight shadings of differences, but in the contrast of brilliant, bold colours of brightness and darkness. In the classroom one sees divergence of performance—an examination of the grades of students at the end of the term usually confirms this, particularly in undergraduate remedial classes. The divergence reported in this dissertation is far greater than that measured by the ability to get the correct answer. It is evidenced in students’ ability to think flexibly—to reverse a direct process; to interpret ambiguous notation; in what they perceive initially and how they categorize their initial perceptions; in the strategies they select; in their abilities to make connections; in the cognitive collages of concept images and schemas they construct and retrieve; and in the confidence they develop in the correctness of their answers or the uncertainty that overwhelms them, leaving them unwilling or unable to take risks in a learning environment different from that they are accustomed to or to renegotiate the didactic contract.

Some students, despite previous experiences which encouraged instrumental learning [Skemp, 1971], are able to develop improved capabilities and deal flexibly and consistently with various representational forms of functions. They develop greater confidence in their ability to do mathematics and acquire confidence and a more positive attitude. Other students are victims of the proceptual divide as Gray and
Tall [1994] have so aptly described them—constrained by their inflexible thinking and strategies—doomed to fail yet again and again and again.

What is it that students are willing and disposed to attend to or to expose? Why is it students enrolled in the same class, with the same instructor and instructional treatment, during the same time period, initially at approximately the same level of competency at the beginning of the semester, take such divergent paths which lead to success or to failure? Why are some students able to think flexibly and others remain inflexible and unchanged? In the next chapter we continue to develop our cognitive collages of MC and SK. The nature of the processes by which they construct knowledge is scrutinized more closely, as we seek answers to these questions.